

THE SUFFICIENCY OF THE MATKOWSKY CONDITION IN THE PROBLEM OF RESONANCE¹

BY
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ABSTRACT. We consider the sufficiency of the Matkowsky condition concerning the differential equation $\epsilon y'' + f(x, \epsilon)y' + g(x, \epsilon)y = 0$ ($-a \leq x \leq b$) under the assumption that $f(0, \epsilon) = 0$ identically in ϵ , $f_x(0, \epsilon) \neq 0$ with $f > 0$ for $x < 0$ and $f < 0$ for $x > 0$. Y. Sibuya proved that the Matkowsky condition implies resonance in the sense of N. Kopell if f and g are convergent power series for $|\epsilon| < \rho$ ($\rho > 0$), $f(x, 0) = -2x$ and the interval $[-a, b]$ is contained in a disc D with center at 0. The main problem in this work is to remove from Sibuya's result the assumption that D is a disc.

1. Introduction. Let us consider the differential equation

$$(1.1) \quad \epsilon y'' + f(x, \epsilon)y' + g(x, \epsilon)y = 0$$

in which the independent variable x ranges over a real interval $[-a, b]$ with $a, b > 0$, and ϵ is a small positive parameter. The coefficients $f(x, \epsilon)$ and $g(x, \epsilon)$ are continuously differentiable real-valued functions of x and ϵ , $f(0, \epsilon) = 0$ identically in ϵ , $f_x(0, \epsilon) \neq 0$ with $f > 0$ for $x < 0$, $f < 0$ for $x > 0$. We consider (1.1) together with boundary conditions

$$(1.2) \quad y(-a, \epsilon) = 1, \quad y(b, \epsilon) = B,$$

where B is a real constant. It is known that, in general, the solution $y(x, \epsilon)$ to (1.1) and (1.2) converges to zero as ϵ tends to zero in the real interval $(-a, b)$. However, some exceptional cases may arise in which the solution $y(x, \epsilon)$ of (1.1) and (1.2) converges on $(-a, b)$ to a nontrivial solution of

$$(1.3) \quad f(x, 0)y' + g(x, 0)y = 0.$$

Indeed, Ackerberg and O'Mally [2] found that unless $l = -g(0, 0)/f_x(0, 0)$ is a nonnegative integer, the solution $y(x, \epsilon)$ of (1.1) and (1.2) converges to zero as ϵ tends to zero, and the term "resonance" is applied to those cases when the limit of $y(x, \epsilon)$ as ϵ tends to zero is a nontrivial solution of (1.3).

Watts [16] considered the problem with $f(x, \epsilon) = -x$ and $g(x, \epsilon) = l + x$, where l is a nonnegative integer, and showed that the above condition is not sufficient for resonance. Cook and Eckhaus [3] found that if $g(x, \epsilon) = l + x + \alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots$,

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then $\alpha_1 = -1$ is also a necessary condition for resonance. They predicted that for resonance we need infinitely many conditions. Therefore one of the major problems of (1.1) and (1.2) is to determine necessary and sufficient conditions for resonance to take place. B. J. Matkowsky [9] proposed the following condition:

“there exists a nontrivial formal power series in ϵ ,

$$y = \sum_{m=0}^{\infty} a_m(x) \epsilon^m,$$

which formally satisfies (1.1) and such that all the $a_m(x)$ are bounded on the real interval $[-a, b]$.

N. Kopell [5] and F. Olver [10] have shown that Matkowsky's condition is necessary for resonance. A formal solution $y = \sum_{m=0}^{\infty} a_m(x) \epsilon^m$ of (1.1) is called an outer expansion. Note that boundary conditions are not involved in the Matkowsky condition.

We say that (1.1) *exhibits resonance in the sense of Kopell on $[-a, b]$ if there exists a solution $y(x, \epsilon)$ satisfying $y(-a, \epsilon) = 1$, (disregarding $y(b, \epsilon) = B$) such that $y(x, \epsilon)$ converges uniformly on $[-a, b]$ to a nontrivial solution of (1.3) as ϵ tends to zero.*

Note. “Resonance” in this paper means “resonance in the sense of N. Kopell”.

MAIN THEOREM. *The Matkowsky condition implies resonance in the sense of Kopell.*

Let

$$L = \epsilon D^2 + fD + g \quad (D = d/dx).$$

Suppose there exists a nontrivial outer expansion $y = \sum_{m=0}^{\infty} a_m(x) \epsilon^m$ whose coefficients $a_m(x)$ are bounded on $[-a, b]$. Set $y = \phi_N + u$ in (1.1), where

$$\phi_N(x, \epsilon) = \sum_{m=0}^N a_m(x) \epsilon^m$$

is a finite sum of the outer expansion. Then

$$(1.4) \quad \epsilon u'' + f(x, \epsilon)u' + g(x, \epsilon)u = -L\phi_N = O(\epsilon^{N+1}).$$

If $u = O(\epsilon^{N+1})$ as ϵ tends to zero, then (1.1) exhibits resonance on $[-a, b]$. However, it is very difficult to show that $u = O(\epsilon^{N+1})$ from (1.4). For example, let us consider

$$(1.5) \quad \epsilon y'' - 2xy' + (p-1)y = \psi(\epsilon),$$

where p is a positive odd integer. The transformations

$$(1.6) \quad y = w \exp(x^2/2\epsilon) \quad \text{and} \quad x = t\epsilon^{1/2}$$

take

$$(1.7) \quad \epsilon y'' - 2xy' + (p-1)y = 0$$

to

$$(1.8) \quad d^2w/dt^2 - (t^2 - p)w = 0$$

(Weber's equation, cf. [17]). P. F. Hsieh and Y. Sibuya [4] constructed the unique solution $w_1(t, p)$ of (1.8) such that $w_1(t, p)$ and $w'_1(t, p)$ admit, respectively, the asymptotic representations

$$(1.9) \quad \begin{aligned} w_1(t, p) &= t^{(p-1)/2} e^{-t^2/2} [1 + O(|t|^{-2})], \\ w'_1(t, p) &= t^{(p+1)/2} e^{-t^2/2} [-1 + O(|t|^{-2})] \end{aligned}$$

as t tends to infinity in any closed subsector of the open sector: $|\arg t| < 3\pi/4$. Let $\omega = \exp(\frac{1}{2}\pi i)$; if t is replaced by $\omega^{-1}t$ and p by $-p$, then (1.8) does not change. Thus $w_1(\omega^{-1}t, -p)$ is also a solution of (1.8) and

$$(1.10) \quad \lim_{t \rightarrow \infty} \omega^{-(p+1)/2} t^{(p+1)/2} e^{-t^2/2} w_1(\omega^{-1}t, -p) = 1, \quad \text{for } \left| \arg t - \frac{\pi}{2} \right| < \frac{3\pi}{4}.$$

Similarly, $w_1(\omega^{-2}t, p)$ and $w_1(\omega t, -p)$ are also solutions of (1.8) with

$$(1.11) \quad \lim_{t \rightarrow \infty} \omega^{-(1-p)/2} t^{(1-p)/2} e^{t^2/2} w_1(\omega^{-2}t, p) = 1, \quad \text{for } |\arg t - \pi| < \frac{3\pi}{4},$$

$$(1.12) \quad \lim_{t \rightarrow \infty} \omega^{(1+p)/2} t^{(1+p)/2} e^{-t^2/2} w_1(\omega t, -p) = 1, \quad \text{for } \left| \arg t + \frac{\pi}{2} \right| < \frac{3\pi}{4}.$$

The general solution of (1.8) is

$$(1.13) \quad w = C_0 w_1(t, p) + C_1 w_1(\omega^{-1}t, -p)$$

where C_0 and C_1 are arbitrary constants. It follows from (1.6), (1.7), (1.8), (1.13) and variation of parameters that (1.5) admits a solution

$$\begin{aligned} y = \exp\left(\frac{x^2}{2\varepsilon}\right) w_1\left(\frac{x}{\sqrt{\varepsilon}}, p\right) &\left\{ C_0 - \int_0^x \frac{w_1(\omega^{-1}\tau/\varepsilon^{1/2}, -p)\psi(\varepsilon)}{2\varepsilon^{1/2}\omega^{-(p-1)/2}\exp(\tau^2/2\varepsilon)} d\tau \right\} \\ &+ \exp\left(\frac{x^2}{2\varepsilon}\right) w_1\left(\frac{\omega^{-1}x}{\sqrt{\varepsilon}}, -p\right) \left\{ C_1 + \int_0^x \frac{w_1(\tau/\varepsilon^{1/2}, p)\psi(\varepsilon)}{2\varepsilon^{1/2}\omega^{-(p-1)/2}\exp(\tau^2/2\varepsilon)} d\tau \right\}. \end{aligned}$$

It follows from (1.9) and (1.10) that

$$\begin{aligned} y \sim \left(\frac{x}{\sqrt{\varepsilon}}\right)^{(p-1)/2} &\left\{ C_0 - \int_0^x \frac{\omega^p \varepsilon^{(p-1)/4} \psi(\varepsilon)}{2\tau^{(p+1)/2}} d\tau \right\} \\ &+ \omega^{(1+p)/2} \varepsilon^{(1+p)/4} x^{-(1+p)/2} \exp\left(\frac{x^2}{\varepsilon}\right) \\ &\times \left\{ C_1 + \int_0^x \frac{\tau^{(p-1)/2} \omega^{(p-1)/2} \psi(\varepsilon)}{2\varepsilon^{(p+1)/4} \exp(\tau^2/\varepsilon)} d\tau \right\} \quad \text{for } x > 0. \end{aligned}$$

Since $\exp(x^2/\varepsilon)$ and $\exp\{(x^2 - \tau^2)/\varepsilon\}$ are exponentially large as ε tends to zero, y is exponentially large as ε tends to zero, even if $\psi(\varepsilon)$ in (1.5) is ε^N . Therefore we obtain $y \neq O(\varepsilon^N)$ as ε tends to zero.

Let us assume that f and g in (1.1) are holomorphic in two complex variables x and ε in the domain $x \in D$, $|\varepsilon| < \rho$, where D is a domain in the x -plane containing the real interval $[-a, b]$ and ρ is a positive number. As we shall show later, *the*

assumption that f and g are convergent power series in ϵ for $|\epsilon| < \rho$ is very important. For example, in the case when

$$f(x, \epsilon) \sim \sum_{m=0}^{\infty} f_m(x) \epsilon^m, \quad g(x, \epsilon) \sim \sum_{m=0}^{\infty} g_m(x) \epsilon^m$$

as ϵ tends to zero and $g(x, \epsilon)$ is not holomorphic in ϵ at $\epsilon = 0$, even if the Matkowsky condition is satisfied, (1.1) may not exhibit resonance. Let us see a more concrete example; consider

$$(1.14) \quad \epsilon y'' - 2xy' + (2 + 2e^{-\alpha/\epsilon})y = 0 \quad (-a \leq x \leq a)$$

where $\alpha > 0$, $\epsilon > 0$ and $a > 0$. Since $e^{-\alpha/\epsilon}$ is asymptotically zero as ϵ tends to zero, (1.14) does have an outer expansion $y = x$, i.e. $a_0(x) = x$ and $a_m(x) = 0$ for $m \geq 1$. Hence, (1.14) satisfies the Matkowsky condition. The general solution of (1.14) is

$$(1.15) \quad y = \exp\left(\frac{x^2}{2\epsilon}\right) \left[C_0 w_1\left(\frac{x}{\epsilon^{1/2}}, 3 + 2e^{-\alpha/\epsilon}\right) + C_1 w_1\left(\frac{\omega^{-1}x}{\epsilon^{1/2}}, -3 - 2e^{-\alpha/\epsilon}\right) \right],$$

where C_0 and C_1 are arbitrary constants. It follows from (1.9) and (1.10) that, for $x > 0$,

$$y(x, \epsilon) \sim C_0(x/\epsilon^{1/2})^{1+e^{-\alpha/\epsilon}} + C_1 \exp(x^2/\epsilon) (\omega^{-1}x/\epsilon^{1/2})^{-2-e^{-\alpha/\epsilon}}.$$

In order to get that $y(x, \epsilon)$ converges uniformly for $x \in [0, a]$ as ϵ tends to zero (y and $\epsilon y'$ being bounded on any closed subinterval of $(-a, a)$), we need to estimate C_0 and C_1 . It follows from (1.15) that

$$\begin{aligned} C_0 = & \left\{ y e^{x^2/2\epsilon} \left[\frac{x}{\epsilon} w_1\left(\frac{\omega^{-1}x}{\epsilon^{1/2}}, -3 - 2e^{-\alpha/\epsilon}\right) \right. \right. \\ & \left. \left. + \omega^{-1}\epsilon^{-1/2} w_1'\left(\frac{\omega^{-1}x}{\epsilon^{1/2}}, -3 - 2e^{-\alpha/\epsilon}\right) \right] \right. \\ & \left. - y' e^{x^2/2\epsilon} w_1\left(\frac{\omega^{-1}x}{\epsilon^{1/2}}, -3 - 2e^{-\alpha/\epsilon}\right) \right\} / 2\epsilon^{-1/2} \omega^{-1-e^{-\alpha/\epsilon}} e^{x^2/\epsilon}, \\ C_1 = & \left\{ -y e^{x^2/2\epsilon} \left[\frac{x}{\epsilon} w_1\left(\frac{x}{\epsilon^{1/2}}, 3 + 2e^{-\alpha/\epsilon}\right) \right. \right. \\ & \left. \left. + \epsilon^{-1/2} w_1'\left(\frac{x}{\epsilon^{1/2}}, 3 + 2e^{-\alpha/\epsilon}\right) \right] \right. \\ & \left. + y' e^{x^2/2\epsilon} w_1\left(\frac{x}{\epsilon^{1/2}}, 3 + 2e^{-\alpha/\epsilon}\right) \right\} / 2\epsilon^{-1/2} \omega^{-1-e^{-\alpha/\epsilon}} e^{x^2/\epsilon}, \end{aligned}$$

hence C_0 and C_1 admit, respectively, the asymptotic representation

$$C_0 \sim \epsilon^{1/2+1/2e^{-\alpha/\epsilon}} \omega^{1+2e^{-\alpha/\epsilon}} \left\{ y x^{-1-e^{-\alpha/\epsilon}} + \frac{1}{2} \epsilon y' x^{-2-e^{-\alpha/\epsilon}} \right\},$$

$$C_1 \sim e^{-(x^2/\epsilon)} \epsilon^{-1-(1/2)e^{-\alpha/\epsilon}} \omega^{-1-e^{-\alpha/\epsilon}} \left\{ -y x^{2+e^{-\alpha/\epsilon}} + \frac{1}{2} \epsilon y' x^{1+e^{-\alpha/\epsilon}} \right\}$$

as ϵ tends to zero and $x \in (0, a)$; therefore we obtain

$$C_0 = C(\epsilon) \epsilon^{1/2}, \quad C_1 = \tilde{C}(\epsilon) e^{-a^2/\epsilon},$$

where $C(\varepsilon)$ is bounded and does not converge to zero as ε tends to zero, and $\tilde{C}(\varepsilon) = O(\varepsilon^{-1})$ as ε tends to zero. Thus,

$$y = \exp\left(\frac{x^2}{2\varepsilon}\right) \left[C(\varepsilon)\varepsilon^{1/2}w_1(x/\varepsilon^{1/2}, 3 + 2e^{-\alpha/\varepsilon}) + \tilde{C}(\varepsilon)e^{-a^2/\varepsilon}w_1(\omega^{-1}x/\varepsilon^{1/2}, -3 - 2e^{-\alpha/\varepsilon}) \right].$$

We now consider $x < 0$. It is known that the function $w_1(x/\varepsilon^{1/2}, 3 + 2e^{-\alpha/\varepsilon})$ admits the following connection formulas (cf. [1, p. 687]):

$$w_1(x/\varepsilon^{1/2}, 3 + 2e^{-\alpha/\varepsilon}) = \frac{\sqrt{2\pi}\omega^{2+e^{-\alpha/\varepsilon}}}{2^{(1/2)(3+2e^{-\alpha/\varepsilon})}\Gamma(-1 - e^{-\alpha/\varepsilon})} \\ \times w_1(\omega^{-1}x/\varepsilon^{1/2}, -3 - 2e^{-\alpha/\varepsilon}) + \omega^{2+2e^{-\alpha/\varepsilon}}w_1(\omega^{-2}x/\varepsilon^{1/2}, 3 + 2e^{-\alpha/\varepsilon}).$$

Also, the inverse of the Gamma function, $\Gamma(-1 - e^{-\alpha/\varepsilon})$, satisfies an estimate

$$K_1e^{-\alpha/\varepsilon} \leq 1/\Gamma(-1 - e^{-\alpha/\varepsilon}) \leq K_2e^{-\alpha/\varepsilon}$$

for some positive numbers K_1 and K_2 ; hence we have

$$y(x, \varepsilon) = \exp\left(\frac{x^2}{2\varepsilon}\right) \left\{ C(\varepsilon)\varepsilon^{1/2}\omega^{2+2e^{-\alpha/\varepsilon}}w_1(\omega^{-2}x/\varepsilon^{1/2}, 3 + 2e^{-\alpha/\varepsilon}) \right. \\ \left. + \left[\frac{\sqrt{2\pi}\omega^{2+e^{-\alpha/\varepsilon}}}{2^{(1/2)(3+2e^{-\alpha/\varepsilon})}\Gamma(-1 - e^{-\alpha/\varepsilon})} C(\varepsilon)\varepsilon^{1/2} + \tilde{C}(\varepsilon)e^{-a^2/\varepsilon} \right] \right. \\ \left. \times w_1(\omega^{-1}x/\varepsilon^{1/2}, -3 - 2e^{-\alpha/\varepsilon}) \right\} \\ \sim C(\varepsilon)\varepsilon^{1/2}(x/\varepsilon^{1/2})^{1+e^{-\alpha/\varepsilon}} + (x/\varepsilon^{1/2})^{-2-e^{-\alpha/\varepsilon}} \\ \times \left[\sqrt{2\pi}2^{-(1/2)(3+2e^{-\alpha/\varepsilon})}\omega^{2e^{-\alpha/\varepsilon}}C(\varepsilon)\varepsilon^{1/2}\exp\left(\frac{x^2 - \alpha}{\varepsilon}\right) \right. \\ \left. + \omega^{2+e^{-\alpha/\varepsilon}}\tilde{C}(\varepsilon)\exp\left\{-\left(\frac{a^2 - x^2}{\varepsilon}\right)\right\} \right].$$

Note that $C(\varepsilon)$ is bounded and does not converge to zero as ε tends to zero, and $\tilde{C}(\varepsilon) = O(\varepsilon^{-1})$ as ε tends to zero. If $\alpha < a^2$ then

$$\sqrt{2\pi}2^{-(3+2e^{-\alpha/\varepsilon})/2}\omega^{2e^{-\alpha/\varepsilon}}C(\varepsilon)\varepsilon^{1/2}e^{(x^2-\alpha)/\varepsilon} + \omega^{2+e^{-\alpha/\varepsilon}}\tilde{C}(\varepsilon)e^{-(a^2-x^2)/\varepsilon}$$

is exponentially large for x near $-a$. Hence (1.14) does not exhibit a resonance if $\alpha < a^2$. Note that if $\alpha \geq a^2$ then (1.14) exhibits resonance on $[-a, a]$.

Y. Sibuya [14] proved that the Matkowsky condition implies resonance if $f(x, 0) = -2x$ and the real interval $[-a, b]$ is contained in a disc D with center at 0. The main ideas in his proof are:

(1) Assume (1.1) satisfies the Matkowsky condition. After making changes of the independent variable and the unknown, (1.1) then becomes

$$(1.16-j) \quad \varepsilon y'' - 2xy' + (p - 1 + \delta_j(\varepsilon))y = 0,$$

respectively, in each of the domains

$$(1.17) \quad x \in D, \quad \varepsilon \in S_j = \{\varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho_0\} \quad (j = 1, 2, \dots, \nu),$$

where p is a positive odd integer, $S_1 \cup S_2 \cup \dots \cup S_\nu = \{\varepsilon: 0 < |\varepsilon| < \rho_0\}$, and

(i) $\delta_j(\varepsilon)$ is holomorphic in $S_j = \{\varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho_0\}$;

(ii) $\delta_j(\varepsilon)$ is asymptotically zero as ε tends to zero in S_j ;

(2) it follows from the connection formulas of Weber's equation (1.8) that

$$|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq M_{jk} \exp(-r^2/|\varepsilon|) \quad \text{for } \varepsilon \in S_j \cap S_k,$$

where M_{jk} is a positive number and r is the radius of disc D . Then by a fundamental lemma in his paper, it follows that $|\delta_j(\varepsilon)| \leq H_j \exp(-r^2/|\varepsilon|)$ for $\varepsilon \in S_j$, where H_j is a positive number.

(3) Since disc D contains the real interval $[-a, b]$, $r^2 \geq \max(a^2, b^2)$. By explanation of the previous example, the estimates of $\delta_j(\varepsilon)$ are good enough for (1.1) to exhibit resonance on $[-a, b]$.

In this paper we consider the general case where D is a domain in the x -plane containing the real interval $[-a, b]$. The main problem is to remove from Sibuya's result the assumption that D is a disc. We will show that there exist transformations taking (1.1) to (1.16- j), respectively, in each of the domains (1.17). In our case, since the domain D contains a disc with radius $r > 0$ and center at 0, we can obtain $|\delta_j(\varepsilon)| \leq H_j \exp(-r^2/|\varepsilon|)$ for $\varepsilon \in S_j$ by using Sibuya's result. However, since the radius r of such a disc is small in general, these estimates of $\delta_j(\varepsilon)$ are not enough for resonance. We shall use cohomological methods and generalizations of the Phragmén-Lindelöf theorem to improve the estimates of $\delta_j(\varepsilon)$. Then we shall show that (1.1) actually exhibits resonance on $[-a, b]$. Information is also taken from the paper by Kreiss and Parter [6].

2. Preliminaries. If we change y by

$$y = w \exp \left\{ -\frac{\varepsilon}{2} \int_0^x [-2\tau + f_1(\tau, \varepsilon)] d\tau \right\},$$

where $f_1(\tau, \varepsilon) = \sum_{m=1}^{\infty} f_m(\tau) \varepsilon^m$, (1.1) is reduced to

$$(2.1) \quad \varepsilon^2 w'' - [x^2 + \varepsilon R(x, \varepsilon)] w = 0,$$

where

$$x^2 + \varepsilon R(x, \varepsilon) = \frac{f(x, \varepsilon)^2}{4} + \varepsilon \left[\frac{1}{2} \frac{\partial f}{\partial x}(x, \varepsilon) - g(x, \varepsilon) \right], \quad R(x, \varepsilon) = \sum_{m=0}^{\infty} R_m(x) \varepsilon^m,$$

and each $R_m(x)$ is holomorphic in the domain D .

(2.1) is equivalent to the system

$$(2.2) \quad \epsilon dW/dx = M(x, \epsilon)W,$$

where

$$W = \begin{bmatrix} w \\ \epsilon dw/dx \end{bmatrix}, \quad M(x, \epsilon) = \begin{bmatrix} 0 & 1 \\ x^2 + \epsilon R(x, \epsilon) & 0 \end{bmatrix}.$$

To prove the main theorem, the following two lemmas are needed, and for their proofs we refer to [14].

LEMMA 2.1. *Let $0 \in \Omega$ be a simply connected domain in D . Then there exists a two-by-two matrix*

$$(2.3) \quad T(x, \epsilon) = \sum_{h=0}^{\infty} T_h(x) \epsilon^h \\ = \begin{bmatrix} A(x, \epsilon) & B(x, \epsilon) \\ \epsilon A'(x, \epsilon) + (x^2 + \epsilon C(\epsilon))B(x, \epsilon) & A(x, \epsilon) + \epsilon B'(x, \epsilon) \end{bmatrix},$$

where $A(x, \epsilon) = \sum_{h=0}^{\infty} A_h(x) \epsilon^h$, $B(x, \epsilon) = \sum_{h=0}^{\infty} B_h(x) \epsilon^h$ and $C(\epsilon) = \sum_{h=0}^{\infty} a_h \epsilon^h$. The components of $T(x, \epsilon)$ are formal power series in ϵ such that:

- (i) the components of the two-by-two matrices $T_h(x)$ are holomorphic in Ω ;
- (ii) $\det T_0(x) = A_0(x)^2 - (xB_0(x))^2 = 1$ in domain Ω ;
- (iii) the formal transformation $W = T(x, \epsilon)V$ reduces (2.2) to

$$(2.4) \quad \epsilon V' = N(x, \epsilon)V$$

where

$$N(x, \epsilon) = \begin{bmatrix} 0 & 1 \\ x^2 + \epsilon C(\epsilon) & 0 \end{bmatrix}, \quad V = \begin{bmatrix} v \\ \epsilon v' \end{bmatrix}$$

in the domain Ω , and

$$(2.5) \quad R_0(0) = a_0.$$

LEMMA 2.2. (1.1) satisfies the Matkowsky condition iff a_0 (cf. (2.5)) is a negative odd integer and $a_m = 0$ ($m \geq 1$).

In order to obtain our main theorem, we will introduce the following notation for a domain in the x -plane.

Let r_3 be a sufficiently small positive number and r_1, r_2 positive numbers with $r_1 > b$, $r_2 > a$, respectively. Set

$$(2.6) \quad D_1 = \left\{ x = x_1 + ix_2 : -r_3 < x_2 < r_3, -\sqrt{r_2^2 - x_2^2} < x_1 < \sqrt{r_1^2 - x_2^2} \right\}$$

(cf. Figure 2.1). Then D_1 is a simply connected domain in the complex x -plane which contains the real interval I . We can choose r_1, r_2 and r_3 so that $D_1 \subset D$.

The main theorem will follow immediately, once the following uniform simplification is proved.

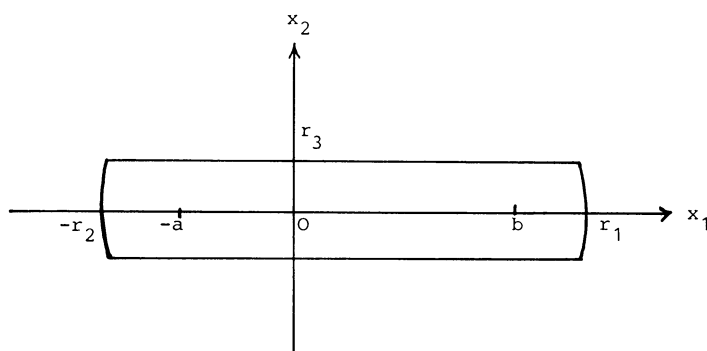


FIGURE 2.1

THEOREM 2.3. Let λ and M be sufficiently small positive numbers and set

$$(2.7) \quad D_2 = D_1 \setminus \{x = re^{i\theta} \in D_1 : -3\pi/4 - \lambda \leq \theta \leq -\pi/4 + \lambda\}$$

(cf. Figure 2.2). Then there exists a two-by-two matrix $S(x, \epsilon)$ whose components are holomorphic in two variables, x and ϵ , in the domain

$$(2.8) \quad x \in D_1,$$

$$(2.9) \quad \{\epsilon : |\arg \epsilon| < \lambda, 0 < |\epsilon| < M\}$$

such that:

(i) the matrix $S(x, \epsilon) \sim T(x, \epsilon)$ (cf. (2.3)) as ϵ tends to zero in (2.9) uniformly in the domain (2.7);

(ii) the transformation $W = S(x, \epsilon)U$ reduces (2.2) to

$$(2.10) \quad \epsilon U' = \begin{bmatrix} 0 & 1 \\ x^2 - \epsilon p & 0 \end{bmatrix} U$$

in the domain (2.8), (2.9).

We shall prove this theorem in §4.C.

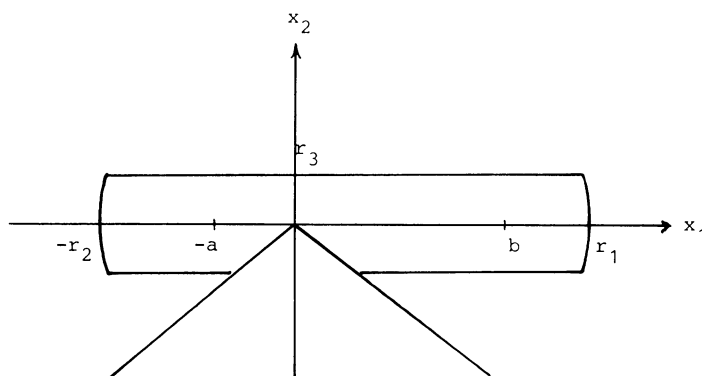


FIGURE 2.2

3. Basic lemmas. Assume that (1.1) satisfies the Matkowsky condition on I . By virtue of Lemma 2.2, this is equivalent to assuming that $a_0 = -p$, where p is a positive odd integer and $a_m = 0$ for $m \geq 1$. Hereafter, we shall assume that $a \leq b$

(otherwise we replace x by $-x$). Manipulating with rotations of domain D_1 and sectors in the ε -plane, we can obtain the following lemma (cf. [8, 14]).

LEMMA 3.1. *There exist sectors $S_j = \{\varepsilon: a_j^* < \arg \varepsilon < b_j^*, 0 < |\varepsilon| < \rho\}$ ($j = 1, 2, \dots, n$), where ρ is a positive number, a^* 's and b^* 's are real numbers, and $S_1 \cup S_2 \cup \dots \cup S_n = \{\varepsilon: 0 < |\varepsilon| < \rho\}$, functions $\delta_1(\varepsilon), \dots, \delta_n(\varepsilon)$ and two-by-two matrices $P_1(x, \varepsilon), \dots, P_n(x, \varepsilon)$ such that:*

- (i) $\delta_j(\varepsilon)$ is holomorphic in S_j ;
- (ii) $\delta_j(\varepsilon)$ is asymptotically zero as ε tends to zero in S_j ;
- (iii) entries of P_j and P_j^{-1} are holomorphic in the domain

(3.1-j)

$$x \in D_1 = \left\{ x = x_1 + ix_2: -r_3 < x_2 < r_3, -\sqrt{r_2^2 - x_2^2} < x_1 < \sqrt{r_1^2 - x_2^2} \right\}, \quad \varepsilon \in S_j,$$

for some domain D_1 in D which contains the real interval I ;

- (iv) P_j (resp. P_j^{-1}) admits the matrix T (resp. T^{-1}) as an asymptotic expansion as ε tends to zero in S_j , which is valid uniformly in D_1 ;
- (v) the transformation $W = P_j(x, \varepsilon)V_j$ reduces (2.2) to

$$(3.1)' \quad \varepsilon V_j' = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon(p + \delta_j(\varepsilon)) & 0 \end{bmatrix} V_j$$

in the domain (3.1-j).

Sibuya [14] proved that the property “asymptotically zero” of $\delta_j(\varepsilon)$ can be replaced by “exponentially small”. In his case the domain of x is a disc with center at 0 and the exponent of ε depends on the radius of the disc. In our case, although the domain of x contains a disc with center at 0, the radius of the disc is small. If we apply his theorem directly, we obtain

$$(3.2) \quad |\delta_j(\varepsilon)| \leq H_1 \exp(-r_3^2/|\varepsilon|) \quad \text{in } S_j, \quad j = 1, 2, \dots, n,$$

for some positive number H_1 .

Let r_0 be a positive number with $a < r_0 < r_2$ and choose ω_0 with $0 < \omega_0 < \frac{1}{2}\pi$ such that $r_3^2 \geq r_0^2 \cos \omega_0 > 0$. Then if $\arg \varepsilon = \omega_0$ (or $-\omega_0$), we get $r_3^2/|\varepsilon| \geq r_0^2 \operatorname{Re}(1/\varepsilon)$. Therefore, for some sector which contains the line segment $\arg \varepsilon = \omega_0$ (or $-\omega_0$), $0 < |\varepsilon| < \beta$, say S_j , we have

$$|\delta_j(\varepsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

for ε on the line segment $\arg \varepsilon = \omega_0$ (or $-\omega_0$), $0 < |\varepsilon| < \rho$. Then we obtain

LEMMA 3.2. *There exist sectors*

$$S_j = \{\varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho\} \quad (j = 1, 2, \dots, v)$$

where

(3.3)

$$-\frac{1}{2}\pi < -\omega_0 = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \dots < a_v < b_{v-1} < b_v = \omega_0 < \frac{1}{2}\pi,$$

functions $\delta_1(\epsilon), \dots, \delta_\nu(\epsilon)$ and two-by-two matrices $P_1(x, \epsilon), \dots, P_\nu(x, \epsilon)$ such that:

(i) $\delta_j(\epsilon)$ is holomorphic in S_j and continuous on

$$S_j^* = \{\epsilon: a_j \leq \arg \epsilon \leq b_j, 0 < |\epsilon| \leq \rho\};$$

(ii) $\delta_j(\epsilon)$ is asymptotically zero as ϵ tends to zero in S_j ;

(iii) $|\delta_1(\epsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\}$ on the line segment $\arg \epsilon = -\omega_0, 0 < |\epsilon| < \rho$;
 $|\delta_\nu(\epsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\}$ on the line segment $\arg \epsilon = \omega_0, 0 < |\epsilon| < \rho$;

(iv) entries of P_j and P_j^{-1} are holomorphic in the domain (3.1-j);

(v) P_j (resp. P_j^{-1}) admits the matrix T (resp. T^{-1}) as an asymptotic expansion as ϵ tends to zero in S_j , which is valid uniformly in D_1 ;

(vi) the transformation $W = P_j(x, \epsilon)V_j$ reduces (2.2) to (3.1)' in the domain (3.1-j).

Note that these functions $\delta_j(\epsilon)$ and sectors S_j are similar to, but slightly different from, those in Lemma 3. Assumption (3.3) means that ν sectors, $S_1, S_2, \dots, S_\nu$, intersect consecutively (i.e. $S_j \cap S_{j+1} \neq \emptyset$), but no three of them intersect.

The estimate of $\delta_1(\epsilon)$ (resp. $\delta_\nu(\epsilon)$) is relatively poor on the ray $\arg \epsilon = \omega_0$ (resp. $\arg \epsilon = -\omega_0$) which is close to the imaginary axis. However, we can substantially improve such an estimate along the real axis by utilizing

THEOREM 3.3. Let S_j and δ_j be as in Lemma 3.2(i), (ii). Assume (3.3) holds and

$$|\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| \leq H_1 \exp\{-\mu \operatorname{Re}(1/\epsilon)\} \quad \text{in } S_j \cap S_{j+1};$$

$|\delta_1(\epsilon)| \leq H_1 \exp\{-\mu \operatorname{Re}(1/\epsilon)\}$ on the line segment $\arg \epsilon = -\omega_0, 0 < |\epsilon| < \rho$; $|\delta_\nu(\epsilon)| \leq H_1 \exp\{-\mu \operatorname{Re}(1/\epsilon)\}$ on the line segment $\arg \epsilon = \omega_0, 0 < |\epsilon| < \rho$, for some positive numbers μ and H_1 .

Then there exists a positive number H such that

$$(3.4) \quad |\delta_j(\epsilon)| \leq H \exp\{-\mu \operatorname{Re}(1/\epsilon)\} \quad \text{in } S_j, \quad j = 1, 2, \dots, \nu.$$

PROOF. See Lin [7].

REMARK. We shall use this theorem with $\mu = r_0^2$. Note that if $\arg \epsilon = 0$ (i.e. ϵ is positive real) and $\epsilon \in S_{j_0}$ for some j_0 , then (3.4) implies $|\delta_{j_0}(\epsilon)| \leq H \exp\{-\mu/\epsilon\}$.

4. Proof of main theorem.

A. Preliminaries. In this section, as an application of Theorem 3.3, we shall derive an estimate

$$(4.1) \quad |\delta_j(\epsilon)| \leq H \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\} \quad \text{for } \epsilon \in S_j,$$

for δ_j and r_0 as in Lemma 3.2. (If necessary we shall replace ρ by $\bar{\rho}$ where $0 < \bar{\rho} < \rho$.) To do this it suffices to prove the estimate

$$(4.2) \quad |\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\} \quad \text{in } S_j \cap S_{j+1}.$$

Note that we already have

$$(4.3) \quad |\delta_1(\epsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\}$$

on the line segment $\arg \epsilon = -\omega_0, 0 < |\epsilon| < \rho$, and

$$(4.4) \quad |\delta_\nu(\epsilon)| \leq H_1 \exp\{-r_0^2 \operatorname{Re}(1/\epsilon)\}$$

on the line segment $\arg \varepsilon = \omega_0$, $0 < |\varepsilon| < \rho$, for some positive number H_1 . To derive (4.2) we shall use

$$(4.5) \quad d^2 w / dt^2 - (t^2 - \alpha)w = 0$$

(Weber's equation, cf. [17]), where α is a complex parameter. Hsieh and Sibuya [4] constructed the unique solution $w_1 = w_1(t, \alpha)$ of (4.5) such that:

- (i) $w_1(t, \alpha)$ is an entire function of (t, α) ;
- (ii) $w_1(t, \alpha)$ and $w_1'(t, \alpha)$ admit, respectively, the asymptotic representations

$$w_1(t, \alpha) = t^{(\alpha-1)/2} e^{-t^2/2} [1 + O(|t|^{-2})],$$

$$w_1'(t, \alpha) = t^{(\alpha+1)/2} e^{-t^2/2} [-1 + O(|t|^{-2})]$$

uniformly on each compact set in the α -space as t tends to infinity in any closed subsector of the open sector $|\arg t| < 3\pi/4$. (Also see Y. Sibuya [12].)

DEFINITION 4.1. *If a solution of (4.5) tends to zero as t tends to infinity along any direction in the sector S , then this solution is said to be subdominant in the sector S .*

Since $\operatorname{Re}(t^2) > 0$ in the sector $|\arg t| < \pi/4$, the solution $w_1(t, \alpha)$ of (4.5) is called a subdominant solution in the sector $|\arg t| < \pi/4$.

Let

$$(4.6) \quad \omega = \exp(\tfrac{1}{2}\pi i).$$

If we replace t by $\omega^{-1}t$ and α by $-\alpha$, then (4.5) does not change. Thus $w_1(\omega^{-1}t, -\alpha)$ is also a solution of (4.5), and

$$\lim_{t \rightarrow \infty} \omega^{-(\alpha+1)/2} t^{(\alpha+1)/2} e^{-t^2/2} w_1(\omega^{-1}t, -\alpha) = 1$$

uniformly in α if α is in a compact set in the α -plane and $|\arg t - \pi/2| < 3\pi/4$. Similarly, $w_1(\omega^{-2}t, \alpha)$ and $w_1(\omega t, -\alpha)$ are also solutions of (4.5) with

$$\lim_{t \rightarrow \infty} \omega^{-(1-\alpha)/2} t^{(1-\alpha)/2} e^{t^2/2} w_1(\omega^{-2}t, \alpha) = 1,$$

$$\lim_{t \rightarrow \infty} \omega^{(1+\alpha)/2} t^{(1+\alpha)/2} e^{-t^2/2} w_1(\omega t, -\alpha) = 1$$

uniformly in α if α is in a compact set in the α -plane and $|\arg t - \pi| < 3\pi/4$, $|\arg t + \pi/2| < 3\pi/4$, respectively.

(4.5) is equivalent to the system

$$(4.7) \quad \frac{d\Psi}{dt} = \begin{bmatrix} 0 & 1 \\ t^2 - \alpha & 0 \end{bmatrix} \Psi,$$

where

$$\Psi = \begin{bmatrix} w \\ dw/dt \end{bmatrix}.$$

Set

$$(4.8-0) \quad \Psi_0(t, \alpha) = \begin{bmatrix} w_1(t, \alpha) & w_1(\omega^{-1}t, -\alpha) \\ \frac{\partial w_1(t, \alpha)}{\partial t} & \omega^{-1} \frac{\partial w_1(\omega^{-1}t, -\alpha)}{\partial t} \end{bmatrix},$$

$$(4.8-1) \quad \Psi_1(t, \alpha) = \begin{bmatrix} w_1(\omega^{-1}t, -\alpha) & w_1(\omega^{-2}t, \alpha) \\ \omega^{-1} \frac{\partial w_1(\omega^{-1}t, -\alpha)}{\partial t} & \omega^{-2} \frac{\partial w_1(\omega^{-2}t, \alpha)}{\partial t} \end{bmatrix},$$

$$(4.8-2) \quad \Psi_2(t, \alpha) = \begin{bmatrix} w_1(\omega^{-2}t, \alpha) & w_1(\omega t, -\alpha) \\ \omega^{-2} \frac{\partial w_1(\omega^{-2}t, \alpha)}{\partial t} & \omega \frac{\partial w_1(\omega t, -\alpha)}{\partial t} \end{bmatrix},$$

and

$$(4.8-(-1)) \quad \Psi_{-1}(t, \alpha) = \begin{bmatrix} w_1(\omega t, -\alpha) & w_1(t, \alpha) \\ \omega \frac{\partial w_1(\omega t, -\alpha)}{\partial t} & \frac{\partial w_1(t, \alpha)}{\partial t} \end{bmatrix}.$$

We consider these matrices of independent solutions of (4.7) in J_0 , J_1 , J_2 and J_{-1} , respectively, where

$$\begin{aligned} J_0 &= \{t: |\arg t - \pi/4| < \pi/2\}, & J_1 &= \{t: |\arg t - \pi/4 - \pi/2| < \pi/2\}, \\ J_2 &= \{t: |\arg t - \pi/4 - \pi| < \pi/2\}, & J_{-1} &= \{t: |\arg t - \pi/4 + \pi/2| < \pi/2\}. \end{aligned}$$

Set

$$(4.9) \quad C(\alpha) = 2^{-\alpha/2} \omega^{(1+\alpha)/2} \sqrt{2\pi} / \Gamma(1/2 - \alpha/2),$$

$$(4.10) \quad \tilde{C}(\alpha) = \omega^{\alpha+3},$$

and

$$(4.11) \quad M(\alpha) = \begin{bmatrix} C(\alpha) & 1 \\ \tilde{C}(\alpha) & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \Psi_0(t, \alpha) &= \Psi_1(t, \alpha)M(\alpha), & \Psi_1(t, \alpha) &= \Psi_2(t, \alpha)M(-\alpha), \\ \Psi_2(t, \alpha) &= \Psi_{-1}(t, \alpha)M(\alpha), & \Psi_{-1}(t, \alpha) &= \Psi_0(t, \alpha)M(-\alpha). \end{aligned}$$

(Cf. Y. Sibuya [14].)

Now we shall derive solutions of the system

$$(4.12-j) \quad \varepsilon V_j'' = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon(p + \delta_j(\varepsilon)) & 0 \end{bmatrix} V_j.$$

Choose a branch of $\varepsilon^{1/2}$ such that $\arg(\varepsilon^{1/2}) = \frac{1}{2} \arg \varepsilon$ for ε in the sector $-\omega_0 < \arg \varepsilon < \omega_0$. Set

$$\Lambda(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{1/2} \end{bmatrix},$$

and

$$\Phi_{jh}(x, \varepsilon) = \Lambda(\varepsilon) \Psi_h(x/\varepsilon^{1/2}, p + \delta_j(\varepsilon)) \quad (h = -1, 0, 1, 2).$$

Then

$$\begin{aligned}\varepsilon \frac{d}{dx} \Phi_{jh}(x, \varepsilon) &= \varepsilon \Lambda(\varepsilon) \frac{d}{dt} \Psi_h\left(\frac{x}{\varepsilon^{1/2}}, p + \delta_j(\varepsilon)\right) \varepsilon^{-1/2} \\ &= \varepsilon^{1/2} \Lambda(\varepsilon) \begin{bmatrix} 0 & 1 \\ \varepsilon^{-1} x^2 - (p + \delta_j(\varepsilon)) & 0 \end{bmatrix} \Lambda(\varepsilon)^{-1} \Phi_{jh}(x, \varepsilon) \\ &= \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon(p + \delta_j(\varepsilon)) & 0 \end{bmatrix} \Phi_{jh}(x, \varepsilon).\end{aligned}$$

Hence, Φ_{jh} are fundamental matrix solutions of (4.12-j) such that

$$\begin{aligned}(4.13-j) \quad \Phi_{j0}(x, \varepsilon) &= \Phi_{j1}(x, \varepsilon) M(p + \delta_j(\varepsilon)), \\ \Phi_{j1}(x, \varepsilon) &= \Phi_{j2}(x, \varepsilon) M(-p - \delta_j(\varepsilon)), \\ \Phi_{j2}(x, \varepsilon) &= \Phi_{j(-1)}(x, \varepsilon) M(p + \delta_j(\varepsilon)), \\ \Phi_{j(-1)}(x, \varepsilon) &= \Phi_{j0}(x, \varepsilon) M(-p - \delta_j(\varepsilon)).\end{aligned}$$

Set

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$Q_{jh}(x, \varepsilon) = \Phi_{jh}(x, \varepsilon) \exp\{(-1)^{\frac{h}{2}} x^2 \varepsilon^{-1} J\} \quad (h = -1, 0, 1, 2).$$

Some of the properties of Q_{jh} are given in the following lemmas due to Sibuya [12].

LEMMA 4.2. *Let R and M_0 be arbitrary but fixed positive numbers and λ a sufficiently small positive number. Then $w = w_1(\omega^{-h}x/\varepsilon^{1/2}, \omega^{-2h}\alpha)$ satisfies*

$$\begin{aligned}(4.14) \quad |w_1(\omega^{-h}x/\varepsilon^{1/2}, \omega^{-2h}\alpha) \exp\{(-1)^h x^2/2\varepsilon\}| &\leq \tilde{c} |\varepsilon|^q, \\ |w'_1(\omega^{-h}x/\varepsilon^{1/2}, \omega^{-2h}\alpha) \exp\{(-1)^h x^2/2\varepsilon\}| &\leq \tilde{c} |\varepsilon|^q\end{aligned}$$

in the domain

$$(4.15) \quad |x| < R, \quad |\arg(x/\varepsilon^{1/2}) - \tfrac{1}{2}h\pi| \leq 3\pi/4 - \lambda, \quad 0 < |\varepsilon| < M_0, \quad |\arg \varepsilon| \leq \omega_0,$$

where $\tilde{c}$ is a positive constant depending on λ , and q is a real number.

LEMMA 4.3. *Let r , R and M_0 be arbitrary but fixed positive numbers and λ a sufficiently small positive number. Suppose $\psi(\varepsilon)$ is a given function of ε which is holomorphic in a sector*

$$(4.16) \quad \{\varepsilon: \rho_1 \leq \arg \varepsilon \leq \rho_2, 0 < |\varepsilon| < M_0\},$$

where $-\omega_0 \leq \rho_1 < \rho_2 \leq \omega_0$ and

$$(4.17) \quad \psi(\varepsilon) \sim 0$$

as ε tends to zero in (4.16). Then

$$\begin{aligned} & \left[w_1 \left(\omega^{-h} \frac{x}{\varepsilon^{1/2}}, \omega^{-2h} \alpha + \psi(\varepsilon) \right) - w_1 \left(\omega^{-h} \frac{x}{\varepsilon^{1/2}}, \omega^{-2h} \alpha \right) \right] \exp \left\{ (-1)^h \frac{x^2}{2\varepsilon} \right\} \sim 0, \\ & \left[w_1' \left(\omega^{-h} \frac{x}{\varepsilon^{1/2}}, \omega^{-2h} \alpha + \psi(\varepsilon) \right) - w_1' \left(\omega^{-h} \frac{x}{\varepsilon^{1/2}}, \omega^{-2h} \alpha \right) \right] \exp \left\{ (-1)^h \frac{x^2}{2\varepsilon} \right\} \sim 0 \end{aligned}$$

uniformly for

$$(4.18) \quad |x| \leq R, \quad |\arg(x/\varepsilon^{1/2}) - h\pi/2| \leq 3\pi/4 - \lambda, \quad |\alpha| \leq r,$$

as ε tends to zero in the sector (4.16).

Given any ε in S_j , by Lemmas 4.2 and 4.3 we know that if (x, ε) is in a domain

$$(4.19-h) \quad x \in D_1, \quad |\arg(x/\varepsilon^{1/2}) - \pi/4 - h\pi/2| \leq \pi/2 - \lambda,$$

where λ is a small positive number, then we have

$$\|Q_{jh}(x, \varepsilon)\| \leq \tilde{c} |\varepsilon|^q, \quad \|Q_{jh}(x, \varepsilon)^{-1}\| \leq \tilde{c} |\varepsilon|^q,$$

where $\tilde{c}$ is a positive number depending on λ , q is a real number and $\|\cdot\|$ denotes the usual matrix norm. Furthermore, the matrix $Q_{jh}(x, \varepsilon) - Q_{j+1,h}(x, \varepsilon)$ is asymptotically zero uniformly in the domain (4.19-h) as ε tends to zero in $S_j \cap S_{j+1}$.

B. Estimate. Let $P_j(x, \varepsilon)$ and $P_{j+1}(x, \varepsilon)$ be the matrices given in Lemma 3.2. Then $P_j(x, \varepsilon)\Phi_{jh}(x, \varepsilon)$ and $P_{j+1}(x, \varepsilon)\Phi_{j+1,h}(x, \varepsilon)$ are two fundamental matrix solutions of (2.2) in the domain $x \in D_1, \varepsilon \in S_j \cap S_{j+1}$. Therefore, there exist two-by-two matrices $L_h(\varepsilon)$ such that

$$(4.20-h) \quad P_{j+1}(x, \varepsilon)\Phi_{j+1,h}(x, \varepsilon) = P_j(x, \varepsilon)\Phi_{jh}(x, \varepsilon)L_h(\varepsilon).$$

Note that $L_h(\varepsilon)$ does not depend on x . It follows from (4.20-0) that

$$\exp \left\{ -\frac{x^2 J}{2\varepsilon} \right\} L_0(\varepsilon) \exp \{x^2 J/2\varepsilon\} = Q_{j0}(x, \varepsilon)^{-1} P_j(x, \varepsilon)^{-1} P_{j+1}(x, \varepsilon) Q_{j+1,0}(x, \varepsilon).$$

Hence, the matrix

$$(4.21) \quad \exp \{-x^2 J/2\varepsilon\} L_0(\varepsilon) \exp \{x^2 J/2\varepsilon\} - I_2$$

is asymptotically zero as ε tends to zero in $S_j \cap S_{j+1}$ uniformly in the domain (4.19-0), where I_2 is the two-by-two identity matrix.

In the same way we can prove that the matrix

$$(4.22) \quad \exp \{x^2 J/2\varepsilon\} L_1(\varepsilon) \exp \{-x^2 J/2\varepsilon\} - I_2$$

is asymptotically zero as ε tends to zero in $S_j \cap S_{j+1}$ uniformly in the domain (4.19-1).

From the connection formulas (4.13-j), (4.20-0) and (4.20-1), we can derive

$$(4.23) \quad L_1(\varepsilon) = M(p + \delta_j(\varepsilon)) L_0(\varepsilon) M(p + \delta_{j+1}(\varepsilon))^{-1}.$$

Set

$$(4.24) \quad L_0(\varepsilon) = \begin{bmatrix} C_{11}(\varepsilon) & C_{12}(\varepsilon) \\ C_{21}(\varepsilon) & C_{22}(\varepsilon) \end{bmatrix}, \quad L_1(\varepsilon) = \begin{bmatrix} \hat{C}_{11}(\varepsilon) & \hat{C}_{12}(\varepsilon) \\ \hat{C}_{21}(\varepsilon) & \hat{C}_{22}(\varepsilon) \end{bmatrix}.$$

Then from (4.23) and (4.24) we derive

(4.25)

$$\begin{aligned}\hat{C}_{11}(\varepsilon) &= C(p + \delta_j(\varepsilon))C_{12}(\varepsilon) + C_{22}(\varepsilon), \\ \hat{C}_{12}(\varepsilon) &= \{C(p + \delta_j(\varepsilon))C_{11}(\varepsilon) + C_{21}(\varepsilon)\} / \tilde{C}(p + \delta_{j+1}(\varepsilon)) \\ &\quad - C(p + \delta_{j+1}(\varepsilon))\{C(p + \delta_j(\varepsilon))C_{12}(\varepsilon) + C_{22}(\varepsilon)\} / \tilde{C}(p + \delta_{j+1}(\varepsilon)), \\ \hat{C}_{21}(\varepsilon) &= \tilde{C}(p + \delta_j(\varepsilon))C_{12}(\varepsilon), \\ \hat{C}_{22}(\varepsilon) &= \{\tilde{C}(p + \delta_j(\varepsilon))C_{11}(\varepsilon)\} / \tilde{C}(p + \delta_{j+1}(\varepsilon)) \\ &\quad - C(p + \delta_{j+1}(\varepsilon))\{\tilde{C}(p + \delta_j(\varepsilon))C_{12}(\varepsilon)\} / \tilde{C}(p + \delta_{j+1}(\varepsilon)).\end{aligned}$$

Since

$$(4.26) \quad \exp\{-x^2 J/2\varepsilon\} L_0(\varepsilon) \exp\{x^2 J/2\varepsilon\} = \begin{bmatrix} C_{11}(\varepsilon) & C_{12}(\varepsilon) \exp(-x^2/\varepsilon) \\ C_{21}(\varepsilon) \exp(x^2/\varepsilon) & C_{22}(\varepsilon) \end{bmatrix},$$

and

$$(4.27) \quad \exp\{x^2 J/2\varepsilon\} L_1(\varepsilon) \exp\{-x^2 J/2\varepsilon\} = \begin{bmatrix} \hat{C}_{11}(\varepsilon) & \hat{C}_{12}(\varepsilon) \exp(x^2/\varepsilon) \\ \hat{C}_{21}(\varepsilon) \exp(-x^2/\varepsilon) & \hat{C}_{22}(\varepsilon) \end{bmatrix},$$

then, from (4.21) and (4.26), we have

$$(4.28) \quad |C_{21}(\varepsilon)| \leq m_{21} |\exp(-x^2/\varepsilon)|,$$

for some small positive number m_{21} , uniformly in the domain (4.19-0). Note that $|\arg(x/\varepsilon^{1/2}) - \frac{1}{4}\pi| \leq \frac{1}{2}\pi - \lambda$ if and only if $\frac{1}{2}\arg \varepsilon - \frac{1}{4}\pi + \lambda \leq \arg x \leq \frac{1}{2}\arg \varepsilon + \frac{3}{4}\pi - \lambda$. If λ is a sufficiently small positive number (e.g. $0 < \lambda < \frac{1}{4}\pi - \frac{1}{2}\omega_0$), then

$$\frac{1}{2}\arg \varepsilon - \frac{1}{4}\pi + \lambda < \frac{1}{2}\arg \varepsilon - \frac{1}{4}\pi + \frac{1}{4}\pi - \frac{1}{2}\omega_0 < 0,$$

and

$$\frac{1}{2}\arg \varepsilon + \frac{3}{4}\pi - \lambda > \frac{1}{2}\arg \varepsilon + \frac{3}{4}\pi - \frac{1}{4}\pi + \frac{1}{2}\omega_0 > 0.$$

Set $x = r_0$. Since $r_0 \in D_1$ and $\arg(r_0) = 0$ (note that $a < r_0 < r_2 \leq r_1$), then r_0 is in the domain (4.19-0) if $|\arg \varepsilon| \leq \omega_0$. Since (4.28) holds uniformly in the domain (4.19-0), we obtain

$$(4.29) \quad |C_{21}(\varepsilon)| \leq m_{21} \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\} \quad \text{for } \varepsilon \in S_j \cap S_{j+1}.$$

Similarly, from (4.22) and (4.27), we have

$$(4.30) \quad |\hat{C}_{12}(\varepsilon)| \leq \hat{m}_{12} |\exp(-x^2/\varepsilon)|,$$

for some small positive number $\hat{m}_{12}$, uniformly in the domain (4.19-1). Note that $|\arg(x/\varepsilon^{1/2}) - \frac{1}{4}\pi - \frac{1}{2}\pi| \leq \frac{1}{2}\pi - \lambda$ iff

$$\frac{1}{2}\arg \varepsilon + \frac{1}{4}\pi + \lambda \leq \arg x \leq \frac{1}{2}\arg \varepsilon + \frac{5}{4}\pi - \lambda.$$

Since

$$\frac{1}{2} \arg \varepsilon + \frac{1}{4} \pi + \lambda < \frac{1}{2} \arg \varepsilon + \frac{1}{4} \pi + \frac{1}{4} \pi - \frac{1}{2} \omega_0 < \frac{1}{2} \pi,$$

and

$$\frac{1}{2} \arg \varepsilon + \frac{5}{4} \pi - \lambda > \frac{1}{2} \arg \varepsilon + \frac{5}{4} \pi - \frac{1}{4} \pi + \frac{1}{2} \omega_0 > \pi,$$

$x = -r_0$ is in the domain (4.19-1), and we obtain

$$(4.31) \quad |\hat{C}_{12}(\varepsilon)| \leq \hat{m}_{12} \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\} \quad \text{for } \varepsilon \in S_j \cap S_{j+1}.$$

Furthermore, as in Sibuya [14, p. 663], we obtain

$$(4.32) \quad |C_{12}(\varepsilon)| \leq m_{12} \exp\{-r_3^2/|\varepsilon|\} \quad \text{for } \varepsilon \in S_j \cap S_{j+1},$$

where m_{12} is a positive number. It follows from (4.25) that

$$\begin{aligned} C(p + \delta_j(\varepsilon))C_{11}(\varepsilon) - C(p + \delta_{j+1}(\varepsilon))\{C(p + \delta_j(\varepsilon))C_{12}(\varepsilon) + C_{22}(\varepsilon)\} \\ = \hat{C}_{12}(\varepsilon)\tilde{C}(p + \delta_{j+1}(\varepsilon)) - C_{21}(\varepsilon). \end{aligned}$$

Since $C_{11}(\varepsilon)$ is asymptotically one as ε tends to zero in $S_j \cap S_{j+1}$ (cf. (4.21) and (4.26)), we can derive (from (4.29) and (4.31))

$$\begin{aligned} \left| C(p + \delta_j(\varepsilon)) - C(p + \delta_{j+1}(\varepsilon)) \left\{ \frac{C_{22}(\varepsilon)}{C_{11}(\varepsilon)} + C(p + \delta_j(\varepsilon)) \frac{C_{12}(\varepsilon)}{C_{11}(\varepsilon)} \right\} \right| \\ \leq m^* \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\} \quad \text{for } \varepsilon \in S_j \cap S_{j+1}, \end{aligned}$$

for some positive number m^* . Let

$$f_j(\varepsilon) = \frac{C_{22}(\varepsilon)}{C_{11}(\varepsilon)} + C(p + \delta_j(\varepsilon)) \frac{C_{12}(\varepsilon)}{C_{11}(\varepsilon)}.$$

Then $f_j(\varepsilon)$ is holomorphic in $S_j \cap S_{j+1}$ and asymptotically one as ε tends to zero in $S_j \cap S_{j+1}$ (cf. (4.21), (4.26) and (4.32)).

LEMMA 4.4. *There exist functions $g_j(\varepsilon)$ in S_j which are, respectively, holomorphic in S_j and asymptotically one as ε tends to zero in S_j such that $f_j(\varepsilon) = g_{j+1}(\varepsilon)/g_j(\varepsilon)$ ($j = 1, 2, \dots, \nu - 1$) in $S_j \cap S_{j+1}$.*

REMARK. Here we replace ρ by a smaller $\tilde{\rho}$.

We shall prove this lemma in §4D. By Lemma 4.4 we obtain

$$(4.33) \quad |C(p + \delta_j(\varepsilon))g_j(\varepsilon) - C(p + \delta_{j+1}(\varepsilon))g_{j+1}(\varepsilon)| \leq m \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

in $S_j \cap S_{j+1}$ for some positive number m . Since $dC(p)/d\alpha \neq 0$ and $C(p) = 0$ (cf. (4.9)), we have

$$(4.34) \quad K_1 |\delta_j(\varepsilon)| \leq |C(p + \delta_j(\varepsilon))| \leq K_2 |\delta_j(\varepsilon)|$$

in S_j for some positive numbers K_1 and K_2 . Let

$$\phi_j(\varepsilon) = C(p + \delta_j(\varepsilon))g_j(\varepsilon) \quad \text{for } \varepsilon \in S_j.$$

Then from (4.33) we have

$$|\phi_j(\varepsilon) - \phi_{j+1}(\varepsilon)| \leq m \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\} \quad \text{in } S_j \cap S_{j+1}.$$

Also, from (4.34), (4.3) and (4.4) we have

$$|\phi_1(\varepsilon)| \leq \tilde{K} \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

on the line segment $\arg \varepsilon = -\omega_0$, $0 < |\varepsilon| < \rho$, and

$$|\phi_v(\varepsilon)| \leq \tilde{K} \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

on the line segment $\arg \varepsilon = \omega_0$, $0 < |\varepsilon| < \rho$ for some positive number $\tilde{K}$. Let $H_1 = \max\{m, \tilde{K}\}$; by Theorem 3.3 we obtain

$$(4.35) \quad |\phi_j(\varepsilon)| \leq H \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

for some positive number H for $\varepsilon \in S_j$, $j = 1, 2, \dots, v$. Thus, from (4.34) and (4.35) we have

$$|\delta_j(\varepsilon)| \leq M_1 \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

for some positive number M_1 for $\varepsilon \in S_j$, $j = 1, 2, \dots, v$. This completes the proof of (4.1).

C. Resonance. In this section we shall prove Theorem 2.3 and derive the resonance from Theorem 2.3. To do this we shrink our domain D_1 so that $a < r_2 < r_0$, where r_0 is the same number as in the last section, and choose a sufficiently small positive number λ such that $\cos \lambda > r_2^2/r_0^2$.

Let us recall that there is a transformation $W = P(x, \varepsilon)V$ which reduces (2.2) to

$$(4.36) \quad \varepsilon V'' = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon(p + \delta(\varepsilon)) & 0 \end{bmatrix} V$$

where $P(x, \varepsilon) \sim T(x, \varepsilon)$ uniformly in the domain D_1 as ε tends to zero in

$$(4.37) \quad \{\varepsilon: |\arg \varepsilon| < \lambda, 0 < |\varepsilon| < M\}.$$

In order to finish the proof of Theorem 2.3, we need to find a two-by-two matrix, say $B(x, \varepsilon)$, such that the transformation $V = B(x, \varepsilon)U$ reduces (4.36) to

$$(2.10) \quad \varepsilon U'' = \begin{bmatrix} 0 & 1 \\ x^2 - \varepsilon p & 0 \end{bmatrix} U$$

in the domain (4.37),

$$(4.38) \quad D_2 = D_1 \setminus \{x = re^{i\theta} \in D_1: -\frac{3}{4}\pi - \lambda \leq \theta \leq -\frac{1}{4}\pi + \lambda\}.$$

Once this is done, we will choose $S(x, \varepsilon)$ in Theorem 2.3 as $S(x, \varepsilon) = P(x, \varepsilon)B(x, \varepsilon)$. Therefore, $B(x, \varepsilon)$ must be asymptotic to I_2 as ε tends to zero in (4.37). Note that from the result in the last section, we already have

$$(4.39) \quad |\delta(\varepsilon)| \leq M_1 \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\}$$

for ε in (4.37) for some positive number M_1 . Theorem 2.3 is an immediate consequence of

THEOREM 4.5. *Let λ and M be sufficiently small positive numbers such that $\cos \lambda > r_2^2/r_0^2$, and let (4.38) hold. Then there exists a two-by-two matrix $B(x, \varepsilon)$ whose components are holomorphic in two variables x and ε in the domain*

$$(4.40) \quad x \in D_1, \quad \varepsilon \in \{\varepsilon: |\arg \varepsilon| < \lambda, 0 < |\varepsilon| < M\}$$

such that:

(i) the matrix $B(x, \varepsilon) \sim I_2$ as ε tends to zero in (4.37) uniformly in the domain (4.38);

(ii) the transformation

$$(4.41) \quad V = B(x, \varepsilon)U$$

reduces (4.36) to (2.10) in the domain (4.40).

PROOF. Set

$$\begin{aligned} \Phi_h(x, \varepsilon) &= \Lambda(\varepsilon)\Psi_h(x/\varepsilon^{1/2}, p + \delta(\varepsilon)) \\ \tilde{\Phi}_h(x, \varepsilon) &= \Lambda(\varepsilon)\Psi_h(x/\varepsilon^{1/2}, p) \end{aligned} \quad (h = 0, 1)$$

(cf. (4.8-0) and (4.8-1)). Then $\Phi_h(x, \varepsilon)$ and $\tilde{\Phi}_h(x, \varepsilon)$ are fundamental matrix solutions of (4.36) and (2.10), respectively, such that

$$\Phi_0(x, \varepsilon) = \Phi_1(x, \varepsilon)M(p + \delta(\varepsilon)) \quad \text{and} \quad \tilde{\Phi}_0(x, \varepsilon) = \tilde{\Phi}_1(x, \varepsilon)M(p)$$

where M is given by (4.11). Set

$$B(x, \varepsilon) = \Phi_0(x, \varepsilon)\tilde{\Phi}_0(x, \varepsilon)^{-1}.$$

Then the transformation $V = B(x, \varepsilon)U$ reduces (4.38) to (2.10). Therefore, the main part of the proof is to show that $B(x, \varepsilon) - I_2$ is asymptotically zero as ε tends to zero in (4.37) uniformly in the domain (4.38).

Set $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$$\begin{cases} Q_h(x, \varepsilon) = \Phi_h(x, \varepsilon)\exp\{(-1)^h x^2 J/2\varepsilon\} \\ \tilde{Q}_h(x, \varepsilon) = \tilde{\Phi}_h(x, \varepsilon)\exp\{(-1)^h x^2 J/2\varepsilon\} \end{cases} \quad (h = 0, 1).$$

By Lemmas 4.2 and 4.3 we know that if ε is in (4.37) and

$$(4.42-h) \quad x \in D_1, \quad |\arg x - \frac{1}{4}\pi - \frac{1}{2}h\pi| \leq \frac{1}{2}\pi - \lambda,$$

where λ is a small positive number, then we have

$$(4.43-h) \quad \begin{cases} \|Q_h(x, \varepsilon)\| \leq \tilde{c}|\varepsilon|^q, & \|Q_h(x, \varepsilon)^{-1}\| \leq \tilde{c}|\varepsilon|^q, \\ \|\tilde{Q}_h(x, \varepsilon)\| \leq \tilde{c}|\varepsilon|^q, & \|\tilde{Q}_h(x, \varepsilon)^{-1}\| \leq \tilde{c}|\varepsilon|^q, \end{cases}$$

where $\tilde{c}$ is a positive number depending on λ and q is a real number. Furthermore, the matrix

$$(4.44-h) \quad Q_h(x, \varepsilon) - \tilde{Q}_h(x, \varepsilon)$$

is asymptotically zero as ε tends to zero in (4.37) uniformly in (4.42-h).

Since $\Phi_1(x, \varepsilon)$ and $B(x, \varepsilon)\tilde{\Phi}_1(x, \varepsilon)$ are two fundamental matrix solutions of (4.36) for $x \in D_1$ and ε is in (4.37), there exists a two-by-two matrix $\tilde{L}_1(\varepsilon)$ such that

$$(4.45) \quad \Phi_1(x, \varepsilon)\tilde{L}_1(\varepsilon) = B(x, \varepsilon)\tilde{\Phi}_1(x, \varepsilon).$$

Note that $\tilde{L}_1(\varepsilon)$ does not depend on x , and

$$\begin{aligned}\tilde{L}_1(\varepsilon) &= \Phi_1(x, \varepsilon)^{-1} B(x, \varepsilon) \tilde{\Phi}_1(x, \varepsilon) \\ &= M(p + \delta(\varepsilon)) \Phi_0(x, \varepsilon)^{-1} B(x, \varepsilon) \tilde{\Phi}_0(x, \varepsilon) M(p)^{-1} \\ &= M(p + \delta(\varepsilon)) M(p)^{-1}.\end{aligned}$$

We have

$$\begin{aligned}M(p + \delta(\varepsilon)) M(p)^{-1} - I_2 &= \begin{bmatrix} C(p + \delta(\varepsilon)) & 1 \\ \tilde{C}(p + \delta(\varepsilon)) & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\tilde{C}(p) \\ 1 & -C(p)/\tilde{C}(p) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \{C(p + \delta(\varepsilon)) - C(p)\}/\tilde{C}(p) \\ 0 & \{\tilde{C}(p + \delta(\varepsilon)) - \tilde{C}(p)\}/\tilde{C}(p) \end{bmatrix} \\ &= \begin{bmatrix} 0 & O(\delta(\varepsilon)) \\ 0 & O(\delta(\varepsilon)) \end{bmatrix} \quad \text{for } |\arg \varepsilon| < \lambda, 0 < |\varepsilon| < M.\end{aligned}$$

Furthermore,

$$\begin{aligned}B(x, \varepsilon) &= \Phi_0(x, \varepsilon) \tilde{\Phi}_0(x, \varepsilon)^{-1} \\ &= Q_0(x, \varepsilon) \exp\{-x^2 J/2\varepsilon\} \exp\{x^2 J/2\varepsilon\} \tilde{Q}_0(x, \varepsilon)^{-1} \\ &= Q_0(x, \varepsilon) \tilde{Q}_0(x, \varepsilon)^{-1}.\end{aligned}$$

It follows from (4.44-0) that $B(x, \varepsilon) - I_2$ is asymptotically zero as ε tends to zero in (4.37) uniformly in the domain (4.42-0). Also, from (4.45) we have

$$\begin{aligned}B(x, \varepsilon) &= \Phi_1(x, \varepsilon) \tilde{L}_1(\varepsilon) \tilde{\Phi}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \exp(x^2 J/2\varepsilon) \tilde{L}_1(\varepsilon) \exp(-x^2 J/2\varepsilon) \tilde{Q}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \cdot \begin{bmatrix} \exp(x^2/2\varepsilon) & 0 \\ 0 & \exp(-x^2/2\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & O(\delta(\varepsilon)) \\ 0 & 1 + O(\delta(\varepsilon)) \end{bmatrix} \\ &\quad \times \exp(-x^2 J/2\varepsilon) \tilde{Q}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \cdot \begin{bmatrix} \exp(x^2/2\varepsilon) & \{\exp(x^2/2\varepsilon)\} O(\delta(\varepsilon)) \\ 0 & \exp(-x^2/2\varepsilon)(1 + O(\delta(\varepsilon))) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \exp(-x^2/2\varepsilon) & 0 \\ 0 & \exp(x^2/2\varepsilon) \end{bmatrix} \cdot \tilde{Q}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \cdot \begin{bmatrix} 1 & \{\exp(x^2/\varepsilon)\} O(\delta(\varepsilon)) \\ 0 & 1 + O(\delta(\varepsilon)) \end{bmatrix} \cdot \tilde{Q}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \left\{ I_2 + \begin{bmatrix} 0 & \exp(x^2/\varepsilon) O(\delta(\varepsilon)) \\ 0 & O(\delta(\varepsilon)) \end{bmatrix} \right\} \tilde{Q}_1(x, \varepsilon)^{-1} \\ &= Q_1(x, \varepsilon) \tilde{Q}_1(x, \varepsilon)^{-1} + Q_1(x, \varepsilon) \begin{bmatrix} 0 & \{\exp(x^2/\varepsilon)\} O(\delta(\varepsilon)) \\ 0 & O(\delta(\varepsilon)) \end{bmatrix} \tilde{Q}_1(x, \varepsilon)^{-1},\end{aligned}$$

as ε tends to zero in (4.37). Since

$$(4.44-1) \quad Q_1(x, \varepsilon) - \tilde{Q}_1(x, \varepsilon)$$

is asymptotically zero as ε tends to zero in (4.37) uniformly in (4.42-1), then there exists a two-by-two matrix $F_1(x, \varepsilon)$ such that $Q_1(x, \varepsilon) = \tilde{Q}_1(x, \varepsilon) + F_1(x, \varepsilon)$ and $F_1(x, \varepsilon)$ is asymptotically zero as ε tends to zero in (4.37) uniformly in (4.42-1). Thus,

$$\begin{aligned} B(x, \varepsilon) - I_2 &= F_1(x, \varepsilon) \tilde{Q}_1(x, \varepsilon)^{-1} \\ &+ Q_1(x, \varepsilon) \begin{bmatrix} 0 & \{\exp(x^2/\varepsilon)\} O(\delta(\varepsilon)) \\ 0 & O(\delta(\varepsilon)) \end{bmatrix} \tilde{Q}_1(x, \varepsilon)^{-1}. \end{aligned}$$

Since $\delta(\varepsilon)$ satisfies (4.39) for $|\arg \varepsilon| < \lambda$, $0 < |\varepsilon| < M$, we have

(i) $\delta(\varepsilon)$ is asymptotically zero as ε tends to zero in (4.37).

Also, if x is in (4.42-1) and ε in (4.37) then

$$\begin{aligned} |\delta(\varepsilon) \exp(x^2/\varepsilon)| &\leq M_1 \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon)\} \exp\{r_2^2/|\varepsilon|\} \\ &= M_1 \exp\{-r_0^2 \operatorname{Re}(1/\varepsilon) + r_2^2/|\varepsilon|\} \\ &= M_1 \exp\{-|\varepsilon|^{-1} [r_0^2 \cos(|\arg \varepsilon|) - r_2^2]\} \end{aligned}$$

since $|\arg \varepsilon| < \lambda < \frac{1}{2}\pi$ implies that $\cos(|\arg \varepsilon|) > \cos \lambda > r_2^2/r_0^2$.

Hence, we have

(ii) $\delta(\varepsilon) \exp(x^2/\varepsilon)$ is asymptotically zero as ε tends to zero in (4.37) uniformly in (4.42-1).

Utilizing (i), (ii) and the fact that $F_1(x, \varepsilon)$ is asymptotically zero as ε tends to zero in (4.37) uniformly in (4.42-1), together with (4.43-1), we obtain $B(x, \varepsilon) - I_2$ is asymptotically zero as ε tends to zero in (4.37) uniformly in the domain (4.42-1). The domain D_2 is the union of (4.42-0) and (4.42-1). Thus we complete the proof of Theorem 4.5.

Now we shall derive the resonance from Theorem 2.3. (2.10) is equivalent to

$$(4.46) \quad \varepsilon^2 u'' - (x^2 - \varepsilon p)u = 0,$$

which admits a solution

$$u(x, \varepsilon) = C w_1(x/\varepsilon^{1/2}, p),$$

where C is an arbitrary constant. Since $(p-1)/2$ is a nonnegative integer,

$$(4.47) \quad u(x, \varepsilon) = C e^{-x^2/2\varepsilon} \left(\frac{x}{\varepsilon^{1/2}} \right)^{(p-1)/2} \left\{ 1 + \alpha_1 \frac{\varepsilon}{x^2} + \alpha_2 \frac{\varepsilon^2}{x^4} + \cdots + \alpha_\nu \frac{\varepsilon^\nu}{x^{2\nu}} \right\},$$

where $\alpha_1, \dots, \alpha_\nu$ are constants, ν is a positive integer such that $2\nu \leq (p-1)/2$, and

(4.48)

$$\varepsilon u'(x, \varepsilon) = C \sqrt{\varepsilon} e^{-x^2/2\varepsilon} \left(\frac{x}{\varepsilon^{1/2}} \right)^{(p+1)/2} \left\{ -1 + \beta_1 \frac{\varepsilon}{x^2} + \beta_2 \frac{\varepsilon^2}{x^4} + \cdots + \beta_\nu \frac{\varepsilon^\nu}{x^{2\nu}} \right\},$$

where $\beta_1, \dots, \beta_\nu$ are constants (cf. Hermite polynomial in [17]). Thus,

$$U = \begin{bmatrix} u \\ \varepsilon u' \end{bmatrix}$$

is a solution of (2.10). Set

$$S(x, \varepsilon) = \begin{bmatrix} S_{11}(x, \varepsilon) & S_{12}(x, \varepsilon) \\ S_{21}(x, \varepsilon) & S_{22}(x, \varepsilon) \end{bmatrix}$$

in Theorem 2.3. Then $S(x, \varepsilon) \sim T(x, \varepsilon)$ as ε tends to zero in (4.37) uniformly in D_2 , and

$$(4.49) \quad w(x, \varepsilon) = S_{11}(x, \varepsilon)u(x, \varepsilon) + S_{12}(x, \varepsilon)\varepsilon u'(x, \varepsilon)$$

is a solution of

$$\varepsilon^2 w'' - (x^2 + \varepsilon R(x, \varepsilon))w = 0.$$

Hence

$$y(x, \varepsilon) = \exp\left\{-\frac{1}{2\varepsilon} \int_0^x [-2\tau + \varepsilon f_1(\tau, \varepsilon)] d\tau\right\} w$$

is a solution of (1.1), where $f_1(\tau, \varepsilon)$ is holomorphic in $\tau \in D_1$, $|\varepsilon| < \rho$. It follows from (4.47)–(4.49) that

$$(4.50) \quad y(x, \varepsilon) = C\varepsilon^{-(p-1)/4} \exp\left\{-\frac{1}{2} \int_0^x f_1(\tau, \varepsilon) d\tau\right\} \\ \times \{S_{11}(x, \varepsilon)[x^{(p-1)/2} + \alpha_1 \varepsilon x^{(p-5)/2} + \dots + \alpha_\nu \varepsilon^\nu x^{q_\nu}] \\ + xS_{12}(x, \varepsilon)[-x^{(p-1)/2} + \beta_1 \varepsilon x^{(p-5)/2} + \dots + \beta_\nu \varepsilon^\nu x^{q_\nu}]\}$$

where q_ν is a nonnegative integer. Set $C = \varepsilon^{(p-1)/4}$. Then (4.50) becomes

$$y(x, \varepsilon) = \exp\left\{-\frac{1}{2} \int_0^x f_1(\tau, \varepsilon) d\tau\right\} \\ \times \{S_{11}(x, \varepsilon)[x^{(p-1)/2} + \alpha_1 \varepsilon x^{(p-5)/2} + \dots + \alpha_\nu \varepsilon^\nu x^{q_\nu}] \\ + xS_{12}(x, \varepsilon)[-x^{(p-1)/2} + \beta_1 \varepsilon x^{(p-5)/2} + \dots + \beta_\nu \varepsilon^\nu x^{q_\nu}]\},$$

and

$$\lim_{\varepsilon \rightarrow 0} y(x, \varepsilon) = \left\{ \exp\left[-\frac{1}{2} \int_0^x f_1(\tau, 0) d\tau\right] \right\} \{A_0(x) - xB(x)\} x^{(p-1)/2}$$

uniformly in $x \in \bar{D}_2$. Since $A_0(x)^2 - (xB(x))^2 = 1$ in D_1 , $\lim_{\varepsilon \rightarrow 0} y(x, \varepsilon)$ is a non-trivial solution of $f(x, 0)y' + g(x, 0)y = 0$.

D. Proof of the lemma. We will prove Lemma 4.4.

PROOF. Since $f_1(\varepsilon)$ is holomorphic in $S_1 \cap S_2$ and asymptotically one in $S_1 \cap S_2$, then $\log f_1(\varepsilon)$ is holomorphic in $\tilde{S}_1 \cap \tilde{S}_2$ and asymptotically zero in $\tilde{S}_1 \cap \tilde{S}_2$, where $\tilde{S}_j = \{\varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \tilde{\rho}\}$ with some $\tilde{\rho}$ such that $0 < \tilde{\rho} < \rho$. For given $\theta > 0$, set

$$V_1 = \{\varepsilon: a_2 + \theta < \arg \varepsilon < b_1 - \theta, 0 < |\varepsilon| < \tilde{\rho}\}.$$

Then $V_1 \subset \tilde{S}_1 \cap \tilde{S}_2$ and $\gamma_1 - \gamma_1^*$ are boundaries of V_1 , where

$$\gamma_1 = \{\varepsilon: \varepsilon = te^{i(a_2 + \theta)}, 0 < t < \tilde{\rho}\} \cup \{\varepsilon: \varepsilon = \tilde{\rho}e^{i\xi}, a_2 + \theta \leq \xi \leq (b_1 - a_2)/2\}$$

and

$$\gamma_1^* = \{\varepsilon: \varepsilon = te^{i(b_1-\theta)}, 0 < t < \tilde{\rho}\} \cup \{\varepsilon: \varepsilon = \tilde{\rho}e^{-i\xi}, (b_1 - a_2)/2 \leq \xi \leq b_1 - \theta\}.$$

By the Cauchy integral formula, we have

$$\log f_1(\varepsilon) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{\log f_1(z)}{z - \varepsilon} dz - \frac{1}{2\pi i} \int_{\gamma_1^*} \frac{\log f_1(z)}{z - \varepsilon} dz.$$

Set

$$\tilde{h}_1(\varepsilon) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{\log f_1(z)}{z - \varepsilon} dz, \quad h_1(\varepsilon) = \frac{1}{2\pi i} \int_{\gamma_1^*} \frac{\log f_1(z)}{z - \varepsilon} dz.$$

Since

$$\frac{1}{z - \varepsilon} = \sum_{m=0}^N z^{-(m+1)} \varepsilon^m + \frac{\varepsilon^{N+1}}{z^{N+1}(z - \varepsilon)}, \quad \text{for all } N = 0, 1, \dots,$$

we have

$$\tilde{h}_1(\varepsilon) = \sum_{m=0}^N \left[\frac{1}{2\pi i} \int_{\gamma_1} z^{-(m+1)} \log f_1(z) dz \right] \varepsilon^m + \frac{\varepsilon^{N+1}}{2\pi i} \int_{\gamma_1} \frac{\log f_1(z) dz}{z^{N+1}(z - \varepsilon)}$$

and

$$\left| \frac{\varepsilon}{2\pi i} \int_{\gamma_1} \frac{\log f_1(z)}{z^{N+1}(z - \varepsilon)} dz \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Therefore

$$\tilde{h}_1(\varepsilon) \sim \sum_{n=0}^{\infty} \tilde{p}_n \varepsilon^n$$

as ε tends to zero in $\hat{S}_2$, where

$$\hat{S}_1 = \{\varepsilon: a_1 < \arg \varepsilon < b_1 - \theta, 0 < |\varepsilon| < \tilde{\beta}\} \subset \tilde{S}_1,$$

and

$$\hat{S}_2 = \{\varepsilon: a_2 + \theta < \arg \varepsilon < b_2, 0 < |\varepsilon| < \tilde{\beta}\} \subset \tilde{S}_2.$$

Note that $\tilde{h}_1(\varepsilon)$ is also holomorphic in $\hat{S}_2$. Similarly,

$$h_1(\varepsilon) \sim \sum_{n=0}^{\infty} p_n \varepsilon^n$$

as ε tends to zero in $\hat{S}_1$, and $h_1(\varepsilon)$ is holomorphic in $\hat{S}_1$. Also, $\log f_1(\varepsilon) = \tilde{h}_1(\varepsilon) - h_1(\varepsilon)$.

Thus,

$$f_1(\varepsilon) = \tilde{H}_1(\varepsilon)/H_1(\varepsilon)$$

where $H_1(\varepsilon) = \exp\{h_1(\varepsilon)\}$ and $\tilde{H}_1(\varepsilon) = \exp\{\tilde{h}_1(\varepsilon)\}$ are holomorphic in $\hat{S}_1$ and $\hat{S}_2$, respectively. Furthermore,

$$H_1(\varepsilon) \sim \sum_{n=0}^{\infty} q_n \varepsilon^n$$

as ϵ tends to zero in $\hat{S}_1$ and

$$\tilde{H}_1(\epsilon) \sim \sum_{n=0}^{\infty} q_n \epsilon^n$$

as ϵ tends to zero in $\hat{S}_2$. Since $f_1(\epsilon)$ is asymptotically one as ϵ tends to zero in $\hat{S}_1 \cap \hat{S}_2$ and $f_1(\epsilon)H_1(\epsilon) = \tilde{H}_1(\epsilon)$ in $\hat{S}_1 \cap \hat{S}_2$, we have $q_n = \tilde{q}_n$ for all $n = 0, 1, 2, \dots$

By a theorem of J. F. Ritt (cf. W. Wasow [15]) there exists a function $F_1(\epsilon)$ which is holomorphic in $\hat{S}_1 \cup \hat{S}_2$, where $F_1(\epsilon) \sim \sum_{n=0}^{\infty} q_n \epsilon^n$ as ϵ tends to zero in $\hat{S}_1 \cup \hat{S}_2$. Set

$$G_1(\epsilon) = H_1(\epsilon)/F_1(\epsilon), \quad \tilde{G}_1(\epsilon) = \tilde{H}_1(\epsilon)/F_1(\epsilon).$$

Then $G_1(\epsilon)$ and $\tilde{G}_1(\epsilon)$ are holomorphic in $\hat{S}_1$ and $\hat{S}_2$ and asymptotically one as ϵ tends to zero in $\hat{S}_1$ and $\hat{S}_2$, respectively. Since $f_1(\epsilon)G_1(\epsilon) = \tilde{G}_1(\epsilon)$, by continuation, we obtain $G_1(\epsilon)$ and $\tilde{G}_1(\epsilon)$ are holomorphic in $\tilde{S}_1$ and $\tilde{S}_2$ and asymptotically one as ϵ tends to zero in $\tilde{S}_1$ and $\tilde{S}_2$, respectively. Since $\tilde{S}_2 \cap \tilde{S}_3 = (\tilde{S}_1 \cup \tilde{S}_2) \cap \tilde{S}_3$ and $f_2(\epsilon)\tilde{G}_1(\epsilon)$ is holomorphic in $\tilde{S}_2 \cap \tilde{S}_3$ and asymptotically one as ϵ tends to zero in $\tilde{S}_2 \cap \tilde{S}_3$, there exist $P_1(\epsilon)$ and $Q_1(\epsilon)$ holomorphic in $\tilde{S}_1 \cup \tilde{S}_2$ and $\tilde{S}_3$ and asymptotically one as ϵ tends to zero in $\tilde{S}_1 \cup \tilde{S}_2$ and $\tilde{S}_3$, respectively, such that

$$f_2(\epsilon)\tilde{G}_1(\epsilon) = Q_1(\epsilon)/P_1(\epsilon).$$

Then

$$f_2(\epsilon) = Q_1(\epsilon)/P_1(\epsilon)\tilde{G}_1(\epsilon).$$

Set

$$f_1(\epsilon) = P_1(\epsilon)\tilde{G}_1(\epsilon)/P_1(\epsilon)G_1(\epsilon).$$

Note that $P_1(\epsilon)G_1(\epsilon)$ and $P_1(\epsilon)\tilde{G}_1(\epsilon)$ are holomorphic in $\tilde{S}_1$ and $\tilde{S}_2$ and asymptotically one as ϵ tends to zero in $\tilde{S}_1$ and $\tilde{S}_2$, respectively. Continuing in this fashion, in a finite number of steps the resultant will meet the requirement of Lemma 4.4.

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